Astrodynamics Lecture Notes, Chap 6; also Air Force Office of Scientific Research TN 60-773 (July 1960).

[3] Sundman, K. F., "Mémoire sur le problème des trois corps," Acta Math. **36**, 105–179 (1912).

[4] Pitkin, E. T., "Integration and optimization of sustained-thrust rocket orbits," Ph.D. Dissertation, Univ. of California at Los Angeles (May 1964).

[5] Ostrowski, A. M., *Solution of Equations and Systems of Equations* (Academic Press, New York, 1960), Chap. 7.

[6] Sperling, H., "Computation of keplerian conic sections," ARS J. **31**, 660–661 (1961).

[7] Battin, R. H., *Astronautical Guidance* (McGraw-Hill Book Co., Inc., New York, 1964), Chaps. 2 and 3.

Free Convection on a Vertical Plate with Concentration Gradients

PHILIP F. DICKSON*
Colorado School of Mines, Golden, Colo.

AND

JOHN J. TRAXLER†
Martin Company, Denver, Colo.

Nomenclature

C_p	=	heat capacity, Btu/lb°R
h	=	heat-transfer coefficient, Btu/ft²-sec°R
H	=	enthalpy, Btu/lb
k	=	thermal conductivity, Btu/ft-sec °R
l	=	boundary-layer dimension in integral method = δ when $\delta = \delta_T = \delta_i$, ft
L	=	length of vertical plate, ft
δ	=	momentum boundary-layer thickness, ft
δ_T	=	thermal boundary-layer thickness, ft
δ_i	=	concentration boundary-layer thickness, ft
N	=	mass flux, lb/ft²-sec
u	=	boundary-layer velocity, fps
u_1	=	velocity outside boundary layer of comparable forced-convective flow, fps
μ	=	viscosity, lb/ft-sec
θ	=	$T - \langle T \rangle$, °R
θ_w	=	$T_w - \langle T \rangle$, °R
τ	=	shear stress, lb/ft-sec
q	=	heat flux, Btu/ft²-sec
T	=	temperature, °R
X_A	=	mass fraction of component A, dimensionless
ψ	=	$X_A - \langle X_A \rangle$, dimensionless
ψ_0	=	$X_{A0} - \langle X_A \rangle$, dimensionless
ν	=	μ/ρ = kinematic viscosity, ft²/sec
ρ	=	density, lb/ft³
β	=	$-(1/\rho)(\partial\rho/\partial T)_{p,X_A}$
ξ	=	$-(1/\rho)(\partial\rho/\partial_{X_A})_{p,T}$
N_{Gr}	=	$g\langle\beta\rangle\, \theta_w z^3/\nu^2$ = Grashof number
$N_{\mathrm{Gr}AB}$	=	$g\langle\xi\rangle\, \psi_0 z^3/\nu^2$ = mass-transfer Grashof number
$N_{\mathrm{Gr}L}$	=	$g\langle\beta\rangle\, \theta_w L^3/\nu^2$ = average Grashof number
$N_{\mathrm{Gr}AB_L}$	=	$g\langle\xi\rangle\, \psi_0 L^3/\nu^2$ = average mass-transfer Grashof number
N_{Pr}	=	$C_p\mu/k$ = Prandtl number
N_{Sc}	=	ν/D_{AB} = Schmidt number

Subscripts

0	=	at interface
⟨ ⟩	=	average
w	=	wall

FREE convection in a fluid arises because of instabilities caused by density differences within the fluid. For a single-component or constant-composition fluid, density varies inversely with temperature. In a system of varying composition, however, density is a function of composition, as well as temperature. A variation in composition within the fluid then will either enhance or retard free convection.

With the use of von Karman's integral methods, Eckert and Jackson[1] and Eckert and Drake[2] obtained expressions for the heat-transfer coefficient for a vertical plate in a constant-composition fluid with turbulent and laminar boundary layers, respectively. For turbulent boundary layers, empirical expressions from forced convection were used for wall shear stress, heat flux, velocity, and temperature profiles.

The purpose of this note is to carry out an extension of the analysis by Eckert et al.[1,2] on a vertical plate for the case of a variable concentration in a binary-component fluid. An approximate method of steady-state solution for a situation such as this is that of von Karman and Pohlhausen. In this integral method, an element of fluid is chosen differentially small in one dimension and of finite length in the other (Fig. 1). Finite length l, as a limit on the integral, is chosen as the largest of the three boundary-layer thickness (hydrodynamic, thermal, and concentration) so that, here, free-stream conditions prevail.

Total mass, energy, and z-directed momentum balances may be given as

$$N_{y=l} + \frac{d}{dz}\int_0^l N_z dy = 0 \tag{1}$$

$$q_w = N_{y=l}\langle H\rangle + \frac{d}{dz}\int_0^l N_z H dy \tag{2}$$

and

$$\langle\rho\rangle \frac{d}{dz}\left[\int_0^l u^2 dy\right] dz = g\langle\rho\rangle dz \int_0^l \langle\beta\rangle(T - \langle T\rangle)\, dy + \left[\int_0 \langle\xi\rangle(X_A - \langle X_A\rangle)dy\right] - \tau_w dz \tag{3}$$

where the equation of state is formed by making a double Taylor series of density in temperature and concentration.[3] Thus

$$\rho = \langle\rho\rangle + (\partial\rho/\partial T)_{\langle T\rangle, \langle X_A\rangle}\,(T - \langle T\rangle) + (\partial\rho/\partial X_A)_{\langle :T\rangle, \langle X_A\rangle}\,(X_A - \langle X_A\rangle) + \ldots$$

To obtain solutions of these equations, functional forms of the dependent variables (u, T, and X_A) must be known or assumed.

Turbulent Free Convection

The forms of the temperature and velocity profiles for turbulent free convection have been determined by experiment and are given by Eckert[1] as

$$u = u_1(y/\delta)^{1/7}[1 - (y/\delta)]^4 \tag{4}$$

and

$$\theta = \theta_w[1 - (y/\delta)^{1/7}] \tag{5}$$

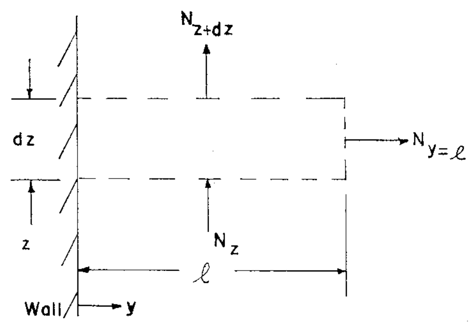

Fig. 1 Boundary-layer element.

Received January 29, 1965; revision received March 31, 1965.
* Assistant Professor.
† Group Engineer, Denver Division. Member AIAA.

where $\delta \approx \delta_T \approx \delta_i$ (which is true for $N_{\mathrm{Pr}} \approx N_{\mathrm{Sc}} \approx 1$). Assuming no mass flux through the vertical wall, and since concentration differences in the y direction will be induced by the flow, it seems reasonable to assume a form for concentration variation in the y direction similar to that for velocity. Therefore

$$\psi = \psi_0(y/\delta)^{1/7}[1 - (y/\delta)]^4 \tag{6}$$

Substitution of Eq. (1) into Eq. (2) for $N_{y=l}$, $C_p(\langle T\rangle - T_0)$ for $(\langle H\rangle - H_0)$, and $\langle\rho\rangle u$ for N_z gives the following for Eq. (2):

$$q_w = \langle\rho\rangle C_p \frac{d}{dz}\int_0^l u\theta\, dy \tag{7}$$

Shearing stress on the wall in turbulent free convection is given empirically[1] by

$$\tau_w = 0.0225\, \rho u_1^2(\nu/u_1\delta)^{1/4} \tag{8}$$

Writing the Reynolds analogy between turbulent exchange of momentum and heat in finite-difference form at the wall gives

$$q_w/\tau_w = C_p(\theta_w/u_1) \tag{9}$$

Substitution for τ_w from Eq. (8) yields

$$q_w = 0.0225\langle\rho\rangle C_p u_1\theta_w(\nu/u_1\delta)^{1/4}N_{\mathrm{Pr}}^{-2/3} \tag{10}$$

where $N_{\mathrm{Pr}}^{-2/3}$ is an empirical correction, which holds for $0.5 < N_{\mathrm{Pr}} < 50$.[1]

Introducing Eqs. (8) and (10) into the energy and momentum equations [Eqs. (3) and (7)] and evaluating the integrals, one obtains

$$0.0523(d/dz)(u_1^2\delta) = 0.125\, g\langle\beta\rangle\theta_w\delta + 0.1463\, \psi_0\langle\xi\rangle\delta - 0.0225\, u_1^2(\nu/u_1\delta)^{1/4} \tag{11}$$

and

$$0.0366(d/dz)(u_1\delta) = 0.0225\, u_1(\nu/u_1\delta)^{1/4}N_{\mathrm{Pr}}^{-2/3} \tag{12}$$

These total differential equations must be solved for u_1 and δ as functions of z. Introducing $u_1 = C_1 z^m$ and $\delta = C_2 z^n$, these equations may be solved in the manner of Eckert[1] to give

$$\delta = \frac{0.565\, z\, N_{\mathrm{Pr}}^{-8/15}(1 + 0.494\, N_{\mathrm{Pr}}^{2/3})^{1/10}}{(N_{\mathrm{Gr}} + 1.17\, N_{\mathrm{Gr}_{AB}})^{1/10}}$$

and

$$u_1 = 1.185(\nu/z)(N_{\mathrm{Gr}} + 1.17\, N_{\mathrm{Gr}_{AB}})^{1/2}/(1 + 0.494\, N_{\mathrm{Pr}}^{2/3})^{1/2}$$

To determine the expression for heat flux at the wall, substitution of these expressions for u_1 and δ into Eq. (10) gives

$$q_w = \frac{0.0295\langle\rho\rangle C_p\theta_w(\nu/z)N_{\mathrm{Pr}}^{-8/15}(N_{\mathrm{Gr}} + 1.17\, N_{\mathrm{Gr}_{AB}})^{2/5}}{(1 + 0.494\, N_{\mathrm{Pr}}^{2/3})^{2/5}}$$

The Nusselt number, a dimensionless surface energy flux, is defined as

$$N_{\mathrm{Nu}L} = hz/k = q_w z/\theta_w k = 0.0295\, N_{\mathrm{Pr}}^{7/15} \times (N_{\mathrm{Gr}} + 1.17\, N_{\mathrm{Gr}_{AB}})^{2/5}/(1 + 0.494\, N_{\mathrm{Pr}}^{2/3})^{2/5}$$

To find the average Nusselt number N_{Nu} for a vertical plate of length L, it is necessary to integrate over L

$$N_{\mathrm{Nu}_L} = \frac{h_L L}{k} = \langle N_{\mathrm{Nu}_z}\rangle = \frac{1}{L}\int_0^L N_{\mathrm{Nu}_z}\, dz = \frac{5}{6}N_{\mathrm{Nu}_z}$$

The average turbulent heat-transfer coefficient for a vertical wall of length L is then

$$h_L = \frac{0.0246\, k/L\, N_{\mathrm{Pr}}^{7/15}(N_{\mathrm{Gr}_L} + 1.17\, N_{\mathrm{Gr}_{AB L}})^{2/5}}{(1 + 0.494\, N_{\mathrm{Pr}}^{2/3})^{2/5}} \tag{13}$$

which reduces to the proper expression when there are no concentration differences (i.e., $N_{\mathrm{Gr}_{AB_L}} = 0$).

Laminar Free Convection

In the laminar case, profiles of the dependent variables are of a different form. Profiles for velocity and temperature in laminar free convection are given by Eckert[2] as

$$u = u_1(y/\delta)[1 - (y/\delta)]^2 \tag{14}$$

and

$$\theta = \theta_w[1 - (y/\delta)]^2 \tag{15}$$

Assume a form for concentration variation in the y direction similar to that for velocity:

$$\psi = \psi_0(y/\delta)[1 - (y/\delta)]^2 \tag{16}$$

From Fourier's law

$$q_w = -k_0(d\theta/dy)_{y=0} = 2\, k_0(\theta_w/\delta)$$

from Eq. (15), and shearing stress on the wall τ_w, in laminar flow, is given by

$$\tau_w = \mu(du/dy)_{y=0} = \mu u_1/\delta$$

from Eq. (14). Introducing these laminar flow expressions for τ_w and q_w into the energy and momentum equations [Eqs. (7) and (3)] and evaluating the integrals, one obtains total differential equations for u_1 and δ in terms of z.

Proceeding as in the turbulent case, the expression for the heat flux at the wall in laminar flow is obtained as

$$q_w = \frac{2\, k_0(\theta_w/\delta) = 0.508\, N_{\mathrm{Pr}}^{1/2}\, k_0(\theta_w/z)(N_{\mathrm{Gr}} + 0.25\, N_{\mathrm{Gr}_{AB}})^{1/4}}{(0.952 + N_{\mathrm{Pr}})^{1/4}}$$

and, defining Nusselt number as a dimensionless surface energy flux, one obtains

$$N_{\mathrm{Nu}_z} = hz/k = q_w\, z/\theta_w k_0 = 0.508\, N_{\mathrm{Pr}}^{1/2}\, (N_{\mathrm{Gr}} + 0.25\, N_{\mathrm{Gr}_{AB}})^{1/4}/(0.952 + N_{\mathrm{Pr}})^{1/4}$$

The average Nusselt number is then given as

$$N_{\mathrm{Nu}L} = \frac{h_L L}{k_0} = \langle N_{\mathrm{Nu}}\rangle = \frac{1}{L}\int_0^L N_{\mathrm{Nu}_z}\, dz = \frac{4}{3}N_{\mathrm{Nu}_z}$$

The average laminar flow heat-transfer coefficient for a vertical wall of length L is then

$$h_L = \frac{0.508\, N_{\mathrm{Pr}}^{1/2}(k_0/L)(N_{\mathrm{Gr}_L} + 0.25\, N_{\mathrm{Gr}_{AB_L}})^{1/4}}{(0.952 + N_{\mathrm{Pr}})^{1/4}}$$

which reduces to the proper expression in the absence of concentration differences (i.e., $N_{\mathrm{Gr}_{AB_L}} = 0$).

Conclusions

The use of integral methods indicates that the effect of a variable composition on laminar and turbulent heat-transfer coefficients can be accounted for by inclusion of a mass-transfer Grashof number. A logical question is raised as to the assumption of the concentration variation through the boundary layer. Integral solutions are known to be fairly insensitive to the profiles chosen and to give correct powers on dimensionless groupings. Coefficients are, however, a function of the profile chosen. To check the validity of Eqs. (6) and (16) and the resultant coefficients on $N_{\mathrm{Gr}_{AB_L}}$, an experimental investigation of this system is planned.

References

[1] Eckert, E. R. G. and Jackson, T. W., "Analysis of turbulent free-convective boundary layer on flat plate," NACA Rept. 1015 (1951).

[2] Eckert, E. R. G. and Drake, R. M., *Heat and Mass Transfer* (McGraw-Hill Book Co., Inc., New York, 1959).

[3] Bird, R. B., Stewart, W. E., and Lightfoot, E. N., *Transport Phenomena* (John Wiley and Sons, Inc., New York, 1960).